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Transmutations That Introduce Group Operators

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Transmutations that relate solutions of the heat and the Euler–Poisson–Darboux equation (EPD) or the equation of generalized axially symmetric potential theory (GASPT) involve a free real parameter. In this paper, it is shown, under suitable commutativity conditions, that this parameter can be replaced by the generator of a continuous group. The relations obtained are used to construct polynomial and fundamental solutions for a variety of group generalizations of the EPD and GASPT equations.

1. INTRODUCTION

A transmutation is an integral transform that connects the solutions of a pair of initial and/or boundary problems in differential equations. There are a variety of ways of constructing such transforms: the method of generalized translations (see [10]), related differential equations [2, 5, 6] and the method of separation of variables [9, 10]. Referring to the second of these, an underlying pair of ordinary differential equations that are related may both contain the same constant while the transform connecting their solutions is independent of this constant. In many such cases, the constant in the pair of differential equations can be replaced by some suitable group or semi-group generator (with accompanying changes in the solution space) and the solutions of the generalized pair of equations thus obtained are connected by the same transform. This approach has been developed extensively and has permitted, for example, the construction of solutions of classical and abstract wave and Dirichlet problems from solutions of abstract heat problems [1, 4, 7]. In other cases, such as the transmutation between solutions of the initial value heat and Euler–Poisson–Darboux (EPD) problems, the connecting transform contains one or more constants present in one of the equations but not in the other [2]. The following questions can then be raised: (i) can one replace the constant(s) introduced by such a transform by some appropriate operator(s) and (ii) what form does the connecting transform then assume?

In this paper, we shall be primarily concerned with these questions for

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transmutations that relate solutions of pairs of abstract hypergeometric equations [2]. While one can consider other equations to demonstrate some of the key ideas (see Sect. 7), the hypergeometric equations have a wide applicability in mathematical physics. For example, the EPD equation and the equation of generalized axially symmetric potential theory (GASPT) can be transformed into hypergeometric type equations. We consider those transforms which introduce only one new constant since repetitions of these methods can be used for treating additional constants and higher order equations.

For these purposes, let X be a Banach space and let $B = B_1^2$ in which B_1 is the infinitesimal generator of a strongly continuous group in X. Then B is the generator of a holomorphic semi-group. Let $\phi \in \mathscr{D}(B^r)$ for r a large positive integer. We consider the following two problems:

$$[D_t - (tD_t + \alpha + 1)\,B]\,v(t) = 0, \quad t > 0; \qquad v(0+) = \phi, \quad \alpha > 0. \tag{1.1a}$$

$$[D_t(tD_t + \beta) - B]\,w(t) = 0, \quad t > 0; \qquad w(0+) = \phi, \quad \beta > 0. \tag{1.1b}$$

If $u(t)$ is a solution of the abstract heat problem

$$u'(t) = Bu(t), \quad t > 0; \qquad u(0+) = \phi, \tag{1.2}$$

then we have (see [2, Theorems 3.1 and 3.2]):

$$v(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-\sigma} \sigma^\alpha u(t\sigma)\, d\sigma, \tag{1.3a}$$

$$w(t) = t^{-\beta} \Gamma(\beta+1)\, L_s^{-1}\{s^{-(\beta+1)} u(1/s)\}_{s \to t^2}. \tag{1.3b}$$

Suppose we attempt to make a formal replacement of the constant α in (1.3a) by some operator A that commutes with B. By rewriting σ^α as $e^{\alpha\lambda}$ with $\lambda = \ln \sigma$, we observe that as σ varies over $(0, \infty)$, then λ varies over $(-\infty, \infty)$. This suggests that σ^A can be assigned a meaning if A is some group generator and motivates the identification of σ^A with $G_A(\ln \sigma)$ in which $G_A(\lambda)$, $-\infty < \lambda < \infty$, denotes the group generated by A. Clearly, the function $u(t\sigma)$ in (1.3a) must be in the domain of A. We also require an interpretation for the gamma function of a group operator if we are to replace α by A in (1.3a). Similar comments can be made about replacing β by such an A in (1.3b).

In Section 2, we review the basic definitions and properties of groups. We define the gamma function of a group generator. This takes the form of an integral operator that involves the group of operators and its existence is tied to the growth properties of the group. Similarly, we define the reciprocal gamma function of a group operator by means of the inverse Laplace transform. Norm estimates for this reciprocal operator can sometimes be obtained when we can call upon the complex inversion formula for the Laplace transform. Even when such norms cannot be estimated, assumptions on the interchange of orders of operations permit the construction of formal solutions which can be verified directly. In Sections 3 and 4, we apply these results to reformulate the transforms (1.3a) and (1.3b) with the parameters α and β replaced by suitable group generators (see [3, 13] for other types of representations involving group operators). A detailed verification of the first of these will be carried out to illustrate how the properties of groups are utilized. We apply these results in Sections 5 and 6 to construct polynomial and other solutions for a variety of "group" generalizations of the EPD and GASPT equations. We also discuss an abstract version of the damped wave equation in Section 7.

2. Groups and Associated Operators

We now recall some basic definitions in connection with continuous groups [8, 14]. Let X be as in the introduction. Then $G(t)$ is said to define a one parameter group of operators in $\mathscr{E}(X)$ if it satisfies the following conditions:

$$G(t_1+t_2)=G(t_1)\cdot G(t_2), \qquad -\infty < t_1, \quad t_2 < \infty, \tag{2.1a}$$

$$G(0)=I, \qquad \text{the identity.} \tag{2.1b}$$

The group $G(t)$ is of class (C_0) if it is continuous at the origin in the strong operator topology. Further, there exist constants $\omega \geqslant 0$, $M_\omega > 0$ such that $\|G(t)\| \leqslant M_\omega e^{\omega t}$. We say that A is the infinitesimal generator for $G(t)$ if

$$A\cdot\phi=\lim_{\tau\to 0}\left(\frac{G(\tau)-I}{\tau}\right)\cdot\phi, \qquad \phi\in\mathscr{D}(A).$$

We refer to A as the generator of the group and denote this group by $G_A(t)$ to indicate its dependence on A. We have $G_A^{(1)}(t)\cdot\varphi=A\cdot G_A(t)\phi$ if $\phi\in\mathscr{D}(A^r)$. If $f(t)$ and $f'(t)\in\mathscr{D}(A^r)$, we also have

$$\{G_A(t)\cdot f(t)\}^{(1)}=A\cdot G_A(t)f(t)+G_A(t)f^{(1)}(t). \tag{2.2}$$

Finally, if A_1 and A_2 are generators of (C_0) groups such that $A_1\cdot A_2=A_2\cdot A_1$ in some appropriate domain, then

$$G_I(t)=e^t, \tag{2.3a}$$

$$G_{\alpha A_1}(t)=G_{A_1}(\alpha t), \qquad \alpha \text{ a real scalar,} \tag{2.3b}$$

$$G_{A_1+A_2}(t)=G_{A_1}(t)\,G_{A_2}(t). \tag{2.3c}$$

Let A denote the generator of a (C_0) group $G_A(t)$. If we identify the scalar function t^α with the group $G_A(\ln t)$ as in the introduction, then $t^{-\alpha}$ gets identified with $G_{-A}(\ln t) = G_A(-\ln t)$ (by (2.3b)).

Finally we observe that

$$\|G_A(\ln t)\,\phi\| \leqslant M_\omega(t^\omega + t^{-\omega}) \cdot \|\phi\|. \tag{2.4}$$

Starting with the scalar definition

$$\Gamma(\alpha+\gamma) = \int_{0+}^{\infty} e^{-t} t^{\alpha+\gamma-1}\, dt = \int_{0+}^{\infty} e^{-t} e^{\alpha \ln t} t^{\gamma-1}\, dt \tag{2.5}$$

and identifying $e^{\alpha \ln t}$ as above, we define, for $\phi \in \mathscr{D}(A)$ and $\gamma > 0$,

$$\Gamma(A+\gamma I)\cdot\phi = \int_{0+}^{\infty} e^{-t} t^{\gamma-1}\{G_A(\ln t)\cdot\phi\}\, dt \tag{2.6}$$

provided that this integral exists in the strong Riemann sense. Taking norms and using (2.4), we have

$$\begin{aligned}\|\Gamma(A+\gamma I)\cdot\phi\| &\leqslant \int_{0+}^{\infty} e^{-t} t^{\gamma-1}\,\|G_A(\ln t)\|\cdot\|\varphi\|\, dt\\ &\leqslant M_\omega \int_{0+}^{\infty} e^{-t} t^{\gamma-1}(t^\omega + t^{-\omega})\cdot\|\phi\|\, dt.\end{aligned}$$

Thus, the integral (2.6) exists if $\gamma > \omega$. For most of our discussion, we assume that $\omega < 1$ which includes the case of the equibounded group.

PROPERTY 1. *Suppose that $\Gamma(A+I)\cdot\phi$ is defined for $\phi \in \mathscr{D}(A^r)$. Then* $(A+I)\cdot\{\Gamma(A+I)\cdot\varphi\} = \Gamma(A+2I)\cdot\varphi$ and, in general, $(A+(\gamma-1)I)$ $\{\Gamma(A+(\gamma-1)I)\cdot\varphi\} = \Gamma(A+\gamma I)\cdot\varphi$.

Proof. From (2.6) and strong integrability, we have

$$(A+I)\cdot\{\Gamma(A+I)\cdot\varphi\} = \int_{0+}^{\infty} e^{-t}\{AG_A(\ln t)\,\varphi\}\, dt + \int_{0+}^{\infty} e^{-t}\{G_A(\ln t)\,\phi\}\, dt. \tag{2.7}$$

But $A\cdot G_A(\ln t)\cdot\varphi = G_A^{(1)}(\ln t)\cdot\varphi = t(d/dt)\{G_A(\ln t)\cdot\varphi\}$, and, by an integration by parts, the first integral in the right member of (2.7) has the evaluation

$$\lim_{\varepsilon\to 0+} te^{-t}\{G_A(\ln t)\cdot\varphi\}\big|_\varepsilon^\infty + \int_{0+}^{\infty} (te^{-t} - e^{-t})\{G_A(\ln t)\cdot\varphi\}\, dt. \tag{2.8}$$

With the growth conditions on the group, the first term in (2.8) vanishes. If we combine the last portion of (2.8) with the last integral in (2.7), we obtain the integral that defines $\Gamma(A+2\cdot I)\cdot\varphi$.

PROPERTY 2. *Suppose that $G_A(\ln t)$ satisfies the condition* (2.4). *Then for $\gamma>\omega$, $\Gamma(A+\gamma I)$ is invertible.*

Proof. Assume that $\Gamma(A+\gamma I)\cdot\varphi=0$. By Property 1, $\Gamma(A+(\gamma+n)I)\varphi=0$ for $n=0,1,2,\ldots$, or

$$\int_{0+}^{\infty} e^{-t}t^{\gamma+n-1}\{G_A(\ln t)\cdot\ \}\,dt=0,\qquad n=0,1,2,\ldots.$$

Multiply this last integral by $(-1)^n s^n/n!$, $s>0$, and sum on n to obtain

$$\int_{0+}^{\infty} e^{-st}e^{-t}t^{\gamma-1}\{G_A(\ln t)\cdot\varphi\}\,dt=0.$$

But the vanishing of this Laplace transform implies that $\{G_A(\ln t)\cdot\varphi\}=0$ for all t. Hence, $\varphi=0$ and the conclusion follows.

If α is real with $\alpha>-1$ and $\gamma\geqslant 1$, then

$$\frac{1}{\Gamma(\alpha+\gamma)}=t^{1-\gamma}\mathscr{L}_s^{-1}\{s^{-\gamma}e^{-\alpha\ln(st)}\}_{s\to t}$$

in which $\mathscr{L}_s^{-1}\{\ \}_{s\to t}$ denotes the inverse Laplace transform with s the variable of the transform and t the variable of inversion. Noting that the left member of this is independent of t, we can replace t by 1 in the right-hand member. Using the same type of identification as earlier, we define for $\gamma>\omega$

$$\begin{aligned}\{\Gamma(A+\gamma I)^{-1}\,\varphi\}&=t^{1-\gamma}\mathscr{L}_s^{-1}[s^{-\gamma}\{G_A(-\ln(st))\cdot\phi]_{s\to t}\\&=\mathscr{L}_s^{-1}[s^{-\gamma}\{G_A(-\ln s)\,\varphi\}]_{s\to 1}.\end{aligned}\tag{2.9}$$

(See [16] for a definition of $1/\Gamma(D_x)$.)

If s is restricted to the positive reals, there is generally no convenient means for estimating $\|\Gamma(A+\gamma I)^{-1}\cdot\varphi\|$. To call upon the complex inversion formula for the Laplace transform imposes restrictions on the choices for $G_A(-\ln s)\cdot\varphi$ and hence A and φ. For example, suppose $A=D_x$ and let $\phi(z)$ be an analytic function that is bounded in the strip $-\pi/2\leqslant I_m(z)\leqslant\pi/2$ with bound $\|\varphi\|$. Then (2.9) becomes

$$\Gamma(D_x+\gamma I)^{-1}\cdot\varphi(x)=\mathscr{L}_s^{-1}[s^{-\gamma}\varphi(x-\ln s)]_{s\to 1}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^s}{\zeta^\gamma}\,\varphi(x-\ln\zeta)\,d\zeta,$$

where $c > 0$. With the change of variables $\zeta = c + i\lambda$, this becomes

$$\Gamma(D_x + \gamma I)^{-1} \cdot \varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{c+i\lambda}}{(c+i\lambda)^{\gamma}} \varphi(x - \ln(c + i\lambda))\, d\lambda. \quad (2.10)$$

Since $c + i\lambda = \rho e^{i\theta}$ with $\rho^2 = c^2 + \lambda^2$ and $\theta = \tan^{-1}(\lambda/c)$, it is easy to check that $|I_m(x - \ln(c + i\lambda))| \leqslant \pi/2$. Hence, $\|\Gamma(D_x + \gamma I)^{-1} \cdot \varphi(x)\| \leqslant (e^c/2\pi) \|\varphi\| \int_0^{\infty} (c^2 + \lambda^2)^{-\gamma/2}\, d\lambda$ which exists for $\gamma > 1$.

If we assume that we can interchange orders of integration, the inversion of Laplace transforms, and the application of group operators, it follows that

$$\begin{aligned}
\Gamma(A+I) \cdot \{\Gamma(A+1)^{-1} \cdot \varphi\} &= \int_{0+}^{\infty} e^{-\xi} G_A(\ln \xi) \{\Gamma(A+I)^{-1} \cdot \varphi\}\, d\xi \\
&= \int_{0+}^{\infty} e^{-\xi} G_A(\ln \xi) \{\mathscr{L}_s^{-1} [s^{-1} G_A(-\ln(st)) \cdot \varphi]_{s \to t}\}\, d\xi \\
&= \int_{0+}^{\infty} e^{-\xi} \left\{ \mathscr{L}_s^{-1} \left[\frac{1}{s} \left\{ G_A \left(\ln \frac{\xi}{st} \right) \cdot \varphi \right\} \right]_{s \to t} \right\} d\xi \\
&= \mathscr{L}_s^{-1} \left\{ \frac{1}{s} \int_{0+}^{\infty} e^{-\xi} G_A \left(\ln \frac{\xi}{st} \right) \cdot \phi\, d\xi \right\}_{s \to t}. \quad (2.11)
\end{aligned}$$

With the change of variables $\xi = s\sigma$ in this last integral (assuming $s > 0$), the last member of (2.11) becomes

$$\mathscr{L}_s^{-1} \left\{ \int_{0+}^{\infty} e^{-s\sigma} \left[G_A \left(\ln \frac{\sigma}{t} \right) \cdot \varphi \right] d\sigma \right\}_{s \to t} = G_A \left(\ln \left(\frac{t}{t} \right) \right) \cdot \phi = G_A(0) \cdot \phi = \phi.$$

We obtain the expected result and this exhibits the consistency of the definitions of $\Gamma(A+I)$ and $\Gamma(A+I)^{-1}$. One can view this demonstration in another way, namely that the consistency of the definitions of these two operators implies the interchanges of orders of operations in (2.11). We shall make use of such interchanges in formulating the "group" generalizations of (1.3a) and (1.3b) in the sections to follow.

Finally, we note a number of "translation" groups that will be used in Sections 5–7. Let $p(x)$ be a nondecreasing function with $p(x) > 0$ for $x \geqslant a$ and consider the initial value problem

$$u_t(x, t) = p(x)\, u_x(x, t), \qquad u(x, 0+) = \varphi(x), \quad \varphi(x) \in C^1. \quad (2.12)$$

With the change of variables $\xi = \int_a^x d\sigma/p(\sigma) = F(x)$, this problem transforms into

$$\tilde{u}_t(\xi, t) = \tilde{u}_\xi(\xi, t), \qquad \tilde{u}(\xi, 0) = \varphi(F^{-1}(\xi))$$

and has the solution $\tilde{u}(\xi, t) = \varphi(F^{-1}(\xi + t))$. Hence

$$u(x, t) = \phi(F^{-1}(F(x) + t). \tag{2.13}$$

Taking $A = p(x) D_x$, the group $G_A(t)$ is then defined by $G_A(t) \cdot \phi = u(x, t)$ as in (2.13). Specific examples of groups are provided in the following:

$p(x)$	$G_A(t)\,\varphi(x)$
1	$\phi(x+t)$
x	$\phi(xe^t)$
$x^\alpha, \alpha < 1$	$\phi[\{x^{1-\alpha} + (1-\alpha)\,t\}^{1/(1-\alpha)}]$.

3. The Group Generalization of (1.3a)

Let X and B taken as in the introduction. We assume that $D = \mathscr{D}(A^{r_1}) \cap \mathscr{D}(B^{r_2})$ is dense in X for r_1, r_2 sufficiently large positive integers and that $A \cdot B \cdot \phi = B \cdot A \cdot \phi$ for $\phi \in D$. We consider the problem (1.2) and the companion problem

$$[D_t - (tD_t + A + I)\,B]\,v(t) = 0, \qquad t > 0, \quad v(0+) = \phi. \tag{3.1}$$

along with the connecting transform

$$v(t) = \mathscr{L}_s^{-1} \left\{ s^{-1} \int_{0+}^{\infty} e^{-\sigma} [G_A(\ln \sigma/s) \cdot u(t\sigma)]\, d\sigma \right\}_{s \to 1}. \tag{3.2}$$

This transform is the generalization of (1.3a) obtained by replacing the scalar α in (1.3a) by the group generator A of a (C_0) group of growth $\omega < 1$ and 1 in the gamma function by I and then employing the interpretations (2.9) and (2.3a). We interpret the integral in braces in (3.2) in the strong Riemann sense. In the following, we make use of the remarks near the end of Section 2 on interchanging orders of operations.

THEOREM 3.1. *Suppose that $u(t)$ satisfies* (1.2) *and that $u(t) \in D$ for all t. Then the function $v(t)$ given by* (3.2) *satisfies the equation in* (3.1).

Proof. Now

$$\frac{\partial}{\partial t} [G_A(\sigma/s)\, u(t\sigma)] = G_A(\ln \sigma/s) \cdot u^{(1)}(t\sigma) \cdot \sigma = \sigma \cdot B \cdot [G_A(\ln \sigma/s) \cdot u(t\sigma)]$$

so that

$$\frac{\partial v(t)}{\partial t} = B \cdot \mathscr{L}_s^{-1} \left\{ s^{-1} \int_{0+}^{\infty} \sigma e^{-\sigma} [G_A(\ln \sigma/s)\, u(t\sigma)]\, d\sigma \right\}_{s \to 1}. \tag{3.3}$$

From the fact that $(t\sigma)\, u^{(1)}(t\sigma) = \sigma u_\sigma(t\sigma)$, we also have

$$\begin{aligned}(tD_t + A + I)\, v(t) &= \mathscr{L}_s^{-1} \left\{ s^{-1} \int_{0+}^{\infty} e^{-\sigma} [\sigma G_A(\ln \sigma/s)\, u_\sigma(t\sigma) \right. \\ &\qquad \left. + A \cdot G_A(\ln \sigma/s)\, u(t\sigma)]\, d\sigma \right\}_{s \to 1} + v(t).\end{aligned} \tag{3.4}$$

But $A \cdot G_A(\ln \sigma/s)\, u(t\sigma) = \sigma[(\partial/\partial\sigma)\{G_A(\ln \sigma/s)\ u(t\sigma)\} - G_A(\ln \sigma/s)\ u_\sigma(t\sigma)]$ (by (2.2)) so that the right member of (3.4) becomes

$$\mathscr{L}_s^{-1} \left\{ s^{-1} \int_{0+}^{\infty} e^{-\sigma} \sigma (\partial/\partial\sigma) [G_A(\ln \sigma/s)\, u(t\sigma)]\, d\sigma \right\}_{s \to 1} + v(t). \tag{3.5}$$

Integrating this by parts and invoking the growth properties of G_A, we obtain

$$\begin{aligned}(tD_t + A + I)\, v(t) &= \mathscr{L}_s^{-1} \{ s^{-1} [\sigma e^{-\sigma} G_A(\ln \sigma/s) \cdot u(t\sigma)]_0^\infty \}_{s \to 1} \\ &\quad - \mathscr{L}_s^{-1} \left\{ s^{-1} \int_{0+}^{\infty} (e^{-\sigma} - \sigma e^{-\sigma}) [G_A(\ln \sigma/s) \cdot u(t\sigma)]\, d\sigma \right\}_{s \to 1} + v(t) \\ &= \mathscr{L}_s^{-1} \left\{ s^{-1} \int_{0+}^{\infty} \sigma e^{-\sigma} [G_A(\ln \sigma/s) \cdot u(t\sigma)]\, d\sigma \right\}_{s \to 1}.\end{aligned} \tag{3.6}$$

A comparison of the last member of (3.6) with the last member of (3.3) shows that the equation in (3.1) is satisfied for $t > 0$.

In order to show that the function $v(t)$ in (3.2) takes on the correct initial data in some sense, we note that

$$\begin{aligned}&\mathscr{L}_s^{-1} \left\{ s^{-1} \int_{0+}^{\infty} e^{-\sigma} [G_A(\ln \sigma/s) \cdot \varphi]\, d\sigma \right\}_{s \to 1} \\ &\quad = \mathscr{L}_s^{-1} \left\{ \int_{0+}^{\infty} e^{-s\tau} [G_A(\ln \tau) \cdot \varphi]\, d\tau \right\}_{s \to 1} \\ &\quad = G_A(\ln 1) \cdot \varphi = \varphi.\end{aligned} \tag{3.7}$$

Even though $\|u(t) - \phi\|_{t \to 0+} \to 0$, our remarks of the previous section on estimating a norm for the reciprocal gamma operator and, correspondingly, the inverse Laplace transform indicate the difficulties in establishing that $\|v(t)-\phi\|_{t\to 0+}\to 0$. Suppose, however, we rewrite (3.2) in the form $v(t)=v_1(t)+v_2(t)$ with

$$v_1(t)=\mathscr{L}_s^{-1}\left\{s^{-1}\int_{0+}^{R} e^{-\sigma}[G_A(\ln\sigma/s)\cdot u(t\sigma)]\,d\sigma\right\}_{s\to 1}, \tag{3.8a}$$

$$v_2(t)=\mathscr{L}_s^{-1}\left\{s^{-1}\int_{R}^{\infty} e^{-\sigma}[G_A(\ln\sigma/s)\cdot u(t\sigma)]\,d\sigma\right\}_{s\to 1}, \tag{3.8b}$$

in which R is a large positive number. For fixed R, we clearly have

$$v_{1,R}=\lim_{t\to 0+} v_1(t)=\mathscr{L}_s^{-1}\left\{s^{-1}\int_0^R e^{-\sigma}[G_A(\ln\sigma/s)\,\phi]\,d\sigma\right\}_{s\to 1},$$

and in view of (3.7), $\lim_{R\to\infty} v_{1,R}=\phi$. We must show that $v_2(t)$ tends to zero as $R\to\infty$. If $u(t)$ is bounded, we can use the estimate (2.4) to show that

$$\left\|\int_R^\infty e^{-\sigma}[G_A(\ln\sigma/s)\cdot u(t\sigma)]\,d\sigma\right\|\leqslant M_\omega\cdot N\int_R^\infty e^{-\sigma}\left\{\left(\frac{\sigma}{s}\right)^{\omega}+\left(\frac{\sigma}{s}\right)^{-\omega}\right\}d\sigma, \tag{3.9}$$

where $0\leqslant\omega<1$ and $N=\operatorname{Sup}\|u(t\sigma)\|$. As $R\to\infty$, the last member of (3.9) tends to zero independent of t. We conclude that $\lim_{R\to\infty} v_2(t)=0$. Hence we have

THEOREM 3.2. *Let B be the generator of an equibounded semi-group and let A be the generator of a continuous group of growth ω, $0\leqslant\omega<1$. Then the function $v(t)$ given by* (3.2) *converges weakly to ϕ as $t\to 0+$.*

We can similarly show that if $u(t)$ has the representation $u(t)=\phi+tu^*(t)$ with $u^*(t)\in D$ and bounded, then $\lim_{t\to 0+} v(t)=\phi$ weakly.

4. The Group Generalization of (1.3b)

The transform in (1.3b) can be inverted to give

$$u(t)=\frac{1}{\Gamma(\beta+1)}\int_0^\infty e^{-\sigma}\sigma^{\beta}w(t\sigma)\,d\sigma. \tag{4.1}$$

Suppose we identify $\beta+1$ with $A+\gamma I$, $\gamma>1$, and, correspondingly, β with $A+(\gamma-1)I$. Using the identifications in Section 2 for the reciprocal gamma operator and σ^A, we obtain a formal connection between the solution of the problem

$$[D_t(tD_t+A+(\gamma-1)I)-B]\cdot w^*(t)=0,\qquad t>0;\quad w^*(0+)=\phi \tag{4.2}$$

and the solution of (1.2), namely,

$$u(t) = \mathscr{L}_s^{-1} \left\{ s^{-\gamma} \int_{0+}^{\infty} e^{-\sigma} \sigma^{\gamma-1} [G_A(\ln \sigma/s)\, w^*(t\sigma)]\, d\sigma \right\}_{s \to 1}. \tag{4.3}$$

A comparison of this with the connecting transform (3.2) shows that they are the same if $\gamma = 1$.

THEOREM 4.1. *Let $w^*(t)$ denote a solution of* (4.2) *with $\omega^*(t) \in \mathscr{D}$ for all $t > 0$. Then the function $u(t)$ given by* (4.3) *satisfies the equation in* (1.2).

Proof. The calculations involved in this are similar to those in the proof of Theorem 3.2 of [2]. The only new ingredient introduced is the differentiation of $[G_A(\ln \sigma/s) \cdot w^*(t\sigma)]$ with respect to σ and this requires using (2.2). This point was illustrated in the proof of Theorem 3.1.

THEOREM 4.2. *Let B be the generator of an equibounded semi-group and let A be the generator of a continuous group of growth ω, $0 \leqslant \omega < 1$. Then the function $u(t)$ converges weakly to ϕ as $t \to 0+$.*

By employing the same technique as was used in [2], we can show that if $u(t)$ satisfies (1.2), then the function $w^*(t)$ obtained by inverting the transform (4.3) satisfies (4.2). This inverse formula is given by

$$w^*(t) = t^{1-\gamma} \cdot \mathscr{L}_s^{-1} \left\{ s^{-\gamma} \int_{0+}^{\infty} e^{-\sigma} \sigma^{\gamma-1} [G_A(\ln \sigma/st) \cdot u(1/s)]\, d\gamma \right\}_{s \to t}. \tag{4.4}$$

5. THE GENERALIZED EPD EQUATION

In this and Section 6 we apply the notions from Sections 1 and 2 to construct certain solutions of a variety of "group" generalizations of the EPD and GASPT equations. Rather than transform the solutions of the corresponding hypergeometric equations into solutions of these equations we make direct use the "heat" equation (1.2).

The initial value problem for the EPD equation is given by

$$U''(t) + \frac{a}{t} U'(t) = BU(t), \qquad U(0+) = \phi, \quad U'(0+) = 0, \quad a \geqslant 0. \tag{5.1}$$

Its solution is related to the solution of (1.2) by means of the transform

$$U(t) = t^{1-a} \Gamma\left(\frac{a+1}{2}\right) \mathscr{L}_s^{-1} \{ s^{-((a+1)/2)} u(1/4s) \}_{s \to t^2} \qquad [6]. \tag{5.2}$$

Suppose we now replace the parameter a in (5.1) by the group generator $2 \cdot A + (2\gamma - 1) I$. Calling upon the replacements noted in Sections 1 and 2 and denoting the corresponding solution function by $E(t)$, it follows that a solution of

$$E''(t) + t^{-1}\{2A + (2\gamma - 1) I\} E'(t) = BE(t), \qquad E(0+) = \varphi, \quad E(0+) = 0 \tag{5.3}$$

is given by

$$E(t) = t^{2-2\gamma} \mathscr{L}_s^{-1} \left\{ s^{-\gamma} \int_0^\infty e^{-\sigma} \sigma^{\gamma-1} \left[G_A \left(\ln \frac{\sigma}{st^2} \right) u \left(\frac{1}{4s} \right) \right] d\sigma \right\}_{s \to t^2}. \tag{5.4}$$

In this, we understand that A and B have the same meanings as in Section 3. We now apply (5.4) to construct solution representations of (5.3) for various choices of A and B.

(i) *Translations.* Let $x = (x_1, \dots, x_n)$, let $B = \Delta_n$ the n-dimensional Laplacian operator, and take $A = \frac{1}{2} \sum_{j=1}^n \alpha_j D_{x_j}$, $\alpha_j \geqslant 0$ and $\gamma = \frac{1}{2}$. Then if $u(x, t)$ is a bounded solution of

$$u_t(x, t) = \Delta_n u(x, t), \qquad t > 0; \quad u(x, 0+) = \phi(x), \tag{5.5}$$

it follows that a solution of

$$E_{tt}(x, t) + t^{-1} \left(\sum_{j=1}^n \alpha_j D_{x_j} \right) E_t(x, t) = \Delta_n E(x, t), \qquad t > 0,$$
$$E(x, 0+) = \phi(x), \qquad E_t(x, 0+) = 0 \tag{5.6}$$

is given by

$$\begin{aligned} E(x, t) &= t \mathscr{L}_s^{-1} \left[s^{-1/2} \int_{0+}^\infty e^{-\sigma} \sigma^{-1/2} \left\{ \prod_{j=1}^n G_{(\alpha_j/2) D_{x_j}} \left(\ln \left(\frac{\sigma}{st^2} \right) \right) u \left(x, \frac{1}{4s} \right) \right\} d\sigma \right]_{s \to t^2} \\ &= t \mathscr{L}_s^{-1} \left\{ s^{-1/2} \int_{0+}^\infty e^{-\sigma} \sigma^{-1/2} \right. \\ &\quad \left. \times u \left(x_1 + \ln \left(\frac{\sigma}{st^2} \right)^{\alpha_1/2}, \dots, x_n + \ln \left(\frac{\sigma}{st^2} \right)^{\alpha_n/2}, \frac{1}{4s} \right) d\sigma \right\}_{s \to t^2}. \end{aligned} \tag{5.7}$$

The last term follows from the second by means of (2.3b) and the translation property of $G_{D_{x_j}}$.

(ii) *A Polynomial Generator.* Suppose we select $B = D_x^2$ in (5.3),

$A = \alpha D_x (\alpha > 0)$, and $\phi(x) = e^{\lambda x}$. A solution of the corresponding heat problem (1.2) is given by $u(x, t) = e^{\lambda x + \lambda^2 t}$ and

$$G_{D_x}\left(\ln\left(\frac{\sigma}{st^2}\right)^{\alpha/2}\right) u\left(x, \frac{1}{4s}\right) = e^{\lambda x + (\lambda^2/4s)}\left(\frac{\sigma}{st^2}\right)^{\alpha\lambda/2}.$$

Inserting this into (5.4) we get

$$E(x, t) = t^{2-2\gamma-\alpha\lambda} e^{\lambda x} \Gamma\left(\gamma + \frac{\alpha\lambda}{2}\right) \mathscr{L}_s^{-1}\left\{\frac{e^{\lambda^2/4s}}{s^{\gamma+(\alpha\lambda/2)}}\right\}_{s \to t^2}.$$

From [15, p. 245], this becomes

$$\begin{aligned} E(x, t) &= \Gamma\left(\gamma + \frac{\alpha\lambda}{2}\right)\left(\frac{2}{\gamma t}\right)^{(\alpha\lambda + 2\gamma - 2)/2} e^{\lambda x} I_{(\alpha\lambda/2)+\gamma-1}(\lambda t) \\ &= e^{\lambda x} \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n}}{2^{2n} n!} \frac{\Gamma(\gamma + (\alpha\lambda/2))}{\Gamma(n + \gamma + (\alpha\lambda/2))}. \end{aligned} \tag{5.8}$$

If we expand the last member of this in powers of λ, the coefficient of $\lambda^n/n!$ is a polynomial solution of

$$\begin{aligned} E_{tt}(x, t) + t^{-1}[\alpha D_x + 2\gamma - 1] E_t(x, t) &= D_x^2(x, t), \qquad t > 0, \\ E(x, 0+) &= x^n, \qquad E_t(x, 0+) = 0. \end{aligned} \tag{5.9}$$

If we denote this polynomial by $P_n(x, t)$, then the first five such polynomials are given by $P_0(x, t) = 1$, $P_1(x, t) = x$, $P_2(x, t) = x^2 + t^2/(2\gamma)$,

$$P_3(x, t) = x^3 + \frac{3xt^2}{2\gamma} - \frac{3\alpha t^2}{4\gamma^2},$$

and

$$P_4(x, t) = x^4 + \frac{3x^2t^2}{\gamma} - \frac{3\alpha x t^2}{\gamma^2} + \frac{3}{2}\frac{\alpha^2 t^2}{\gamma^3} + \frac{3}{4}\frac{t^4}{\gamma(\gamma+1)}.$$

(iii) *A Fundamental-type representation.* Suppose we select x and Δ_n as in (i) and consider the initial value problem

$$\begin{aligned} E_{tt}(x, y, t) + t^{-1}(2yD_y + (2\gamma - 1) I) E_t(x, y, t) &= \Delta_n E(x, y, t), \\ E(x, y, 0+) = \phi(x, y), \qquad E_t(x, y, 0+) &= 0. \end{aligned} \tag{5.10}$$

In this, $A = yD_y$. The solution of the heat problem

$$u_t(x, y, t) = \Delta_n u(x, y, t), \qquad t > 0; \quad u(x, y, 0) = \varphi(x, y) \tag{5.11}$$

(with y regarded as a parameter) is given by

$$u(x, y, t) = (4\pi t)^{-n/2} \int_{R_n} e^{-r^2(x,\xi)/4t} \varphi(\xi, y)\, d\xi$$

with $r^2(x, \xi) = \sum_{j=1}^{n} (x_j - \xi_j)^2$ and $d\xi = d\xi_1 \cdots d\xi_n$. Using the second entry in the table at the end of Section 2, we have

$$G\left(\ln \frac{\sigma}{st^2}\right) u(x, y, 1/4s) = \left(\frac{s}{\pi}\right)^{n/2} \int_{R_n} e^{-r^2(x,\xi)s} \varphi\left(\xi, \frac{y\sigma}{st^2}\right) d\xi.$$

Employing (5.4) along with the change of variables $\sigma = st^2 z$, we get

$$E(x, y, t) = \frac{t^2}{\pi^{n/2}} \int_{R_n} \int_0^{\infty} z^{\gamma-1} \varphi(\xi, yz) \{\mathscr{L}_s^{-1} [e^{-s(t^2 z + r^2(x,\xi))} s^{n/2}]_{s \to t^2}\}\, dz\, d\xi. \tag{5.12}$$

Suppose, for example, that $n = 2m - 1$, $m = 1, 2, \ldots$. Then

$$\mathscr{L}_s^{-1} \left\{ s^m \cdot \frac{1}{s^{1/2}} e^{-s(t^2 z + r^2(x,\xi))} \right\}_{s \to t^2} = \left(\frac{\partial}{\partial \tau}\right)^m \frac{[\tau - (r^2(x, \xi) + t^2 z)]_+^{-1/2}}{\sqrt{\pi}},$$

where $\tau = t^2$ and $[a - b]_+ = a - b$ if $a > b$ and 0, otherwise. Finally

$$E(x, y, t) = \frac{t^2 (D_\tau)^m}{\pi^m} \int_{B_{2m-1}(x, \tau^{1/2})} \varphi(\xi, yz) \times \left\{ \int_0^{(\tau - r^2)/t^2} z^{\gamma-1} [\tau - r^2(x, \xi) - t^2 z]^{-1/2}\, dz \right\} d\xi \tag{5.13}$$

with τ replaced by t^2 after the differentiation with respect to τ has been carried out. In this $B_{2m-1}(x, \tau^{1/2})$ denotes the $(2m - 1)$-dimensional ball centered at x with radius $\tau^{1/2}$. We can similarly derive a solution of (5.10) if $n = 2m$, $m = 1, 2, \ldots$.

6. The Generalized GASPT Equation

The standard abstract Dirichlet problem for the GASPT equation is given by

$$U_{tt}(t) + \frac{a}{t} U_t(t) + BU(t) = 0, \qquad t > 0, \quad U(0+) = \phi; \quad a < 1. \tag{6.1}$$

Its solution is related to the solution of (1.2) by means of the transform

$$U(t)=\frac{t^{1-a}}{\Gamma\left(\frac{1-a}{2}\right)}\int_0^{\infty} e^{-\sigma t^2}\sigma^{-((a+1)/2)}u(1/4\sigma)\,d\sigma. \tag{6.2}$$

(See [2].) A group generalization of (6.1) can be obtained by replacing the parameter a by $-2A-(2\gamma-1)I$. Once again, if we use the results of Sections 1 and 2 with this replacement, we find that a solution of the problem

$$\begin{aligned} V_{tt}(t)+t^{-1}\{-2A-(2\gamma-1)I\}\,V_t(t)+V(t)=0, \qquad & t>0,\\ V(0+)=\phi \qquad & \end{aligned} \tag{6.3}$$

is given by

$$V(t)=t^{2\gamma}\mathscr{L}_s^{-1}\left\{s^{-\gamma}\int_{0+}^{\infty}\sigma^{\gamma-1}e^{-\sigma t^2}\left[G_A\left(\ln\frac{\sigma t^2}{s}\right)u\left(\frac{1}{4\sigma}\right)\right]d\sigma\right\}_{s\to 1}. \tag{6.4}$$

As an example of the use of (6.4), we construct a solution of the Dirichlet problem

$$\begin{aligned} V_{tt}(x,y,t)-\frac{2y}{t}V_{yt}(x,y,t)+V_{xx}(x,y,t)=0,\\ V(x,y,0+)=\phi(x,y). \end{aligned} \tag{6.5}$$

In (6.4), select $\gamma=\frac{1}{2}$ and $A=yD_y$. Just as in example 3 of Section 5 with $n=1$, we have

$$G_A\left(\ln\frac{\sigma t^2}{s}\right)\cdot u\left(x,y,\frac{1}{4\sigma}\right)=\frac{\sigma^{1/2}}{\pi^{1/2}}\int_{-\infty}^{\infty}e^{-(x-\xi)^2\cdot\sigma}\phi\left(\xi,\frac{y\sigma t^2}{s}\right)d\xi.$$

Inserting this in (6.4) and making the change of variables $\sigma=s\lambda/t^2$, we get

$$\begin{aligned} V(x,y,t)=\frac{1}{\sqrt{\pi}\,t}\int_{-\infty}^{\infty}\int_0^{\infty}\phi(\xi,y\lambda)\\ \times\left\{\mathscr{L}_s^{-1}\left[s^{1/2}e^{-s\lambda}\left\{\frac{t^2+(x-\xi)^2}{t^2}\right\}\right]_{s\to 1}\right\}d\lambda\,d\xi. \end{aligned} \tag{6.6}$$

But

$$\mathscr{L}_s^{-1}\left\{s\cdot\frac{1}{s^{1/2}}e^{-\alpha s}\right\}_{s\to\tau}=\frac{\partial}{\partial\tau}\frac{(\tau-\alpha)_+^{-1/2}}{\sqrt{\pi}}.$$

Introducing this back into (6.6), we get

$$V(x,y,t) = \frac{1}{\pi}\left(\frac{\partial}{\partial\tau}\right)\int_{-\infty}^{\infty}\int_{0}^{\tau t^2/(t^2+(x-\xi)^2)} \frac{\varphi(\xi, y\lambda)}{[\tau t^2 - \lambda(t^2+(x-\xi)^2)]^{1/2}}\, d\lambda\, d\xi \Bigg|_{\tau=1}. \tag{6.7}$$

If ϕ is independent of y, it is an easy exercise to show that (6.7) reduces to the standard Poisson formula for the upper half plane Dirichlet problem.

7. The Damped Wave Equation

In [5], it was shown that if $u(t)$ is a solution of (1.2), then a solution of

$$\begin{aligned} W_{tt}(t) + 2\alpha W_t(t) &= BW(t), & t > 0,\\ W(0+) &= 0, & W_t(0+) = \phi \end{aligned} \tag{7.1}$$

is given by

$$W(t) = \frac{\sqrt{\pi}}{2}\, e^{-\alpha t}\mathscr{L}_s^{-1}\{s^{-3/2}e^{\alpha^2/4s}u(1/4s)\}_{s\to t^2}. \tag{7.2}$$

Suppose we now replace α by the generator of a continuous group A in (7.1). Then A^2 generates a holomorphic semi-group and we can identify $e^{\alpha^2/4s}$ in (7.2) with the semi-group of operators $T_{A^2}(1/4s)$. Hence, we have

Theorem 7.1. *Suppose that A is the generator of a continuous group and that B is the generator of a holomorphic semi-group in X. Suppose that $A \cdot B = B \cdot A$ and $D = \mathscr{D}(A^{r_1}) \cap \mathscr{D}(B^{r_2})$ is dense in X for r_1, r_2 arbitrarily large. Then a solution of*

$$\begin{aligned} \mathscr{W}_{tt}(t) + 2A\mathscr{W}_t(t) &= B\mathscr{W}(t), & t > 0,\\ \mathscr{W}(0+) &= 0, & \mathscr{W}_t(0+) = \varphi, \quad \varphi \in D, \end{aligned} \tag{7.3}$$

is given by

$$\mathscr{W}(t) = \sqrt{\pi}/2)\, G_A(-t)\, \mathscr{L}_s^{-1}\{s^{-3/2}T_{A^2}(1/4s)\cdot u(1/4s)\}_{s\to t^2}. \tag{7.4}$$

Consider the problem

$$\begin{aligned} w_{tt}(x,y,t) + w_{yt}(x,y,t) &= w_{xx}(x,y,t), & t > 0,\\ w(x,y,0+) = 0, \qquad w_t(x,y,0+) &= \varphi(x,y), & \varphi \in C^1. \end{aligned} \tag{7.5}$$

in which $A = \frac{1}{2}D_y$. In this case

$$T_{A^2}(1/4s) \cdot \psi(y) = \frac{2\sqrt{s}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-4(y-\eta)^2 s} \psi(n)\, d\eta$$

for $\psi(y)$ bounded and continuous. Then, employing the same types of computational methods that were used in Sections 5 and 6, it is not difficult to show that a solution of (7.5) is given by

$$w(x, y, t) = \frac{1}{\pi} G_{D_y}\left(-\frac{t}{2}\right) \tilde{W}(x, y, t) = \frac{1}{\pi} \tilde{W}\left(x, y - \frac{t}{2}, t\right) \tag{7.6}$$

in which

$$\tilde{W}(x, y, t) = \int_{\tilde{E}(x,y,t)} \frac{\varphi(\xi, \eta)}{\sqrt{t^2 - (x-\xi)^2 - 4(y-\eta)^2}}\, d\xi\, d\eta \tag{7.7}$$

in which $\tilde{E}(x, y; t)$ denotes the solid ellipse with center (x, y), semi-major axis t, and semi-minor axis $t/2$.

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